Web Site of Robert John Morton

Neural Networks: MLP Non-linear Transfer Function

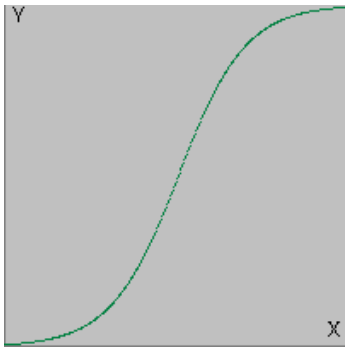
The formula, *look-up table*, *linear interpolation*, *new version of Sigmoid()*, *prototyping in QuickBASIC*, *the 'C' version*, *SigGen()'s coding*, *Sigmoid()'s coding*, *test results*.

The Sigmoid Function

The sigmoid function is so-called because it is shaped like one form of the Greek letter Sigma. Its purpose, within an artificial neuron, is to generate a degree of non-linearity between the neuron's input and output.

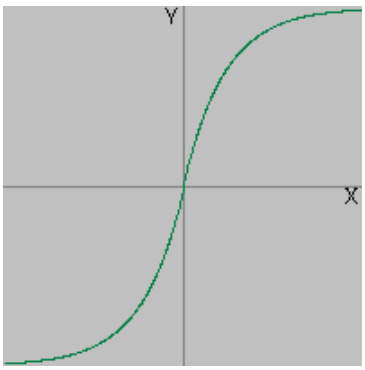
The adjacent applet creates a sigmoid curve by iterating the difference equation y += k(1 − y). This operates over the range 0 to +1 for both input and output. However, perhaps the most familiar way of creating a sigmoid curve is with the natural exponential function:

$$f(x) = \frac{1}{1 + e^{-kx}}$$



If you are using Microsoft Windows and a Security Warning box pops up saying that the application has been blocked from running because it is "untrusted" please click here. If you get a warning message with Linux, please click here.

The basic formula for the sigmoid function above caters only for 'real' values of x from −1 to +1 and produces an output f(x) over the range 0 to +1.

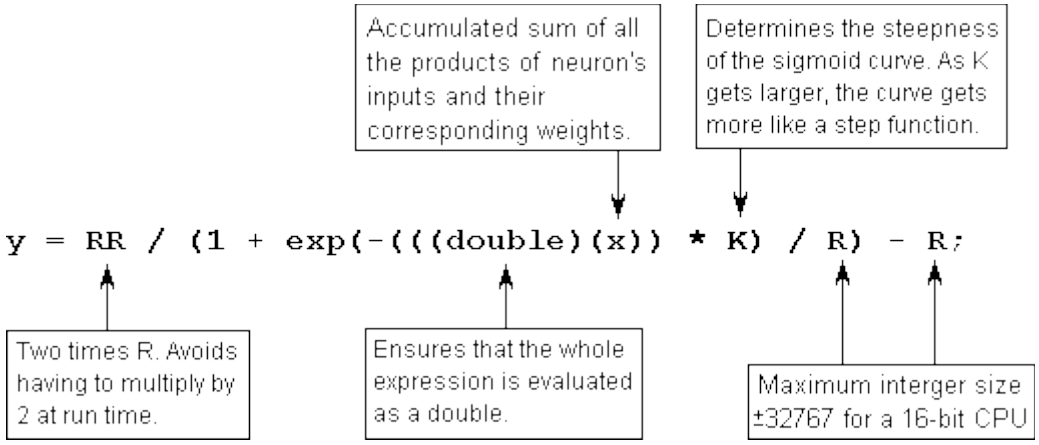


The enhanced version below provides a bipolar output over the same range as the input.

$$f(x) = 32767 \left(\frac{2}{1 + e^{-kx\ /\ 32767}} - 1 \right)$$

This version of the formula also rescales the input and the output to a range of −32767 to +32767 which is the range of the 16-bit integer native to the machine used on this project.

This formula can be written as a 'C' statement as follows:



Since we need only 16-bit precision in the output, y can be a 16-bit **int**. The exp() function takes a floating point double argument, and would therefore force the result of the multiplication and division to a floating point double. However, in order to preserve the 16-bit precision of the input through the calculation, we must cast x to a floating point **double** before multiplying it by K and subsequently dividing it by R.

We shall now build this formula into a 'C' function we shall call Sigmoid().

```
int Sigmoid(int x) {
  #define R  32767   //max value of a signed 16-bit interger
  #define RR 65534   //double it to avoid run-time doubling
  #define K      8   //suitable value for K
  return(RR / (1 + exp( -(((double)(x)) * K) / R) - R);
}
```

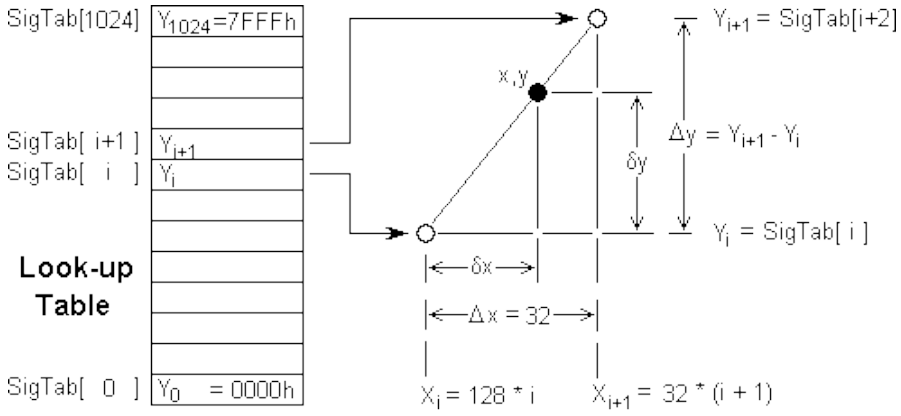
Unfortunately, this function was found to take 5·4 milliseconds to execute on an 80286 PC. Since it is at the very time-critical centre of a network, this is unacceptably slow. A very small increase in speed is possible by placing the formula directly into the neural network function as a single statement, but the gain is almost imperceptible. A better solution is vital.

Look-up Table

A vast speed improvement would obviously be achieved if all 32768 plots covering the range of the 16-bit input and output values were placed in a look-up table. However this would require 64k of memory to hold the table which is also unacceptable. A compromise is to have a smaller table and use linear interpolation to find values of f(x) for values of x in between two of the spot values of x for which table entries have been made.

Linear Interpolation

This works on the bold assumption that any section of the sigmoid curve, between two successive tabulated plots, is a straight line. We know it is not, but we hope that the real curve does not bend too far away from our straight line and thereby keep the error within acceptable limits. Assuming the line is straight enough, we can find the value of f(x) corresponding to an inter-plot value of x by simple geometry as follows:



$$f(x) = \Delta y \frac{\delta x}{128}$$

The table is chosen to have 1025 entries. This is one more than the range of values 0 to 1023 inclusive since the method needs a final plot beyond the extremity of the range to act as a reference for the last valid plot.

NewVersion of Sigmoid()

To make best use of 16-bit interger precision, this is implemented as a table or array containing the 1025 values of f(x) which correspond to 1025 values of x which are multiples of 32. Thus x ranges from 0 to 32768 ie: 0, 32, 64, 96, 128, ... 32736, 32767. The final entry is set to 32767 instead of 32768 to avoid overflow.

So firstly we need a function to create the spot values of f(x) for the table which is stored in the array of short (16-bit) integers SigTab[ ]:

```
short SigTab[1025];  /* values of f(x) for x values
                        which are multiples of 32 */
void SigGen(void) {
  #define R  32767    //maximum extent of both x and f(x)
  #define RR 65556    //65534 + 22
  for(int i = 0; i < 1024; i++)
    SigTab[i] = (double)(
      RR / (1 + exp( -((double)(((long)(i)) << 8)) / R)) - R
    );
  SigTab[1024] = R;
}
```

The left shift of 8 on the long casting of i multiplies i by 32 to get x and then by 8 to get kx (for k = 8). This function with its time-consuming exp()call need be executed once-only during the network's initialisation phase. So now instead of using the slow exp() function we can find a neurone's output from the sum of its weighted inputs by calling the following function:

```
int Sigmoid(int x) {
  int s, y, j;
  if((s = x) < 0) x = -x;   //reverse sign of x if x negative
  y = *(SigTab + (j = x >> 5));
  y += ((*(SigTab + j + 1) - y) * (x & 0x1F)) >> 5;
  if (s < 0) y = -y; //reverse sign of f(x) is x was negative
  return(y);
}
```

This function returns the required value of f(x) in less than 21µs. This is 256 times as fast as the formula. It only disagrees with the formula in the least significant of the 16 bits in only 40% of cases (a maximum error of 30 parts per million). The 256-fold speed gain is merely at the expense of an extra 2k of memory to hold the look-up table. Speeds were measured on an IBM PS/2 Model 30-286 (without an 80287 math co-processor).

Prototyping in QuickBASIC

The thinking behind the Look-up and Interpolation version of Sigmoid() was first validated by prototyping it in Microsoft QuickBASIC. Sigmoid itself was rewritten in QuickBASIC within an exerciser program as follows:

```
DefInt a - x
dim SigTab(1024)
R!  = 32767
RR! = 65534
screen 9 : color ,1   'switch to EGA enhanced mode graphics
'set up suitable screen co-ordinates
window (-48000,-48000) - (+48000,+48000)
line(-32767, 0) - (32767, 0) , 7
line(0, -32767) - (0, 32767) , 7
locate  2,13
print"NEURAL TRANSFER FUNCTION FOR A MULTI-LAYER PERCEPTRON"
locate  4,39 : print"f(x)"
locate  5,34 : print"+32767"
locate 21,42 : print"-32767"
locate 12,66 : print"+32767"
locate 14,10 : print"-32767"
locate 13,70 : print"x"
locate  8,12 : print"SIGMOID FUNCTION"
locate 16,45 : print"f(x) = 2/(1+exp(-k*x)) - 1"
locate 17,45 : print"x = neuron activation level"
locate 18,45 : print"f(x) = neuron output level"
k =  4 : c = 11 : gosub SigGen : gosub SigTest
k =  8 : c = 10 : gosub SigGen : gosub SigTest
k = 16 : c = 12 : gosub SigGen : gosub SigTest
k = 32 : c = 14 : gosub SigGen : gosub SigTest
LOCATE 24,1:print"HIT ANY KEY TO QUIT";
E: x$ = inkey$ : if x$ = "" goto E
end

SigGen:
  for i = 0 to 1023
    w! = i
    v! = RR! / (1 + exp(-w! * 32 * k / R!)) - R!
    y = v! : SigTab(i) = y
  next
  SigTab(1024) = R!
return

SigTest:
  k$ = str$(k) : k$ = "k =" + space$(3 - len(k$)) + k$
  locate 11-q,56: print k$ : q=q+1
  x = -32767 : start! = TIMER
' locate 17,5:print"Errors ="
Label:
  gosub Sigmoid : pset (x, y), c
```

```
  if x < 32767 then x = x + 1 : goto Label
  locate 23,4:print"TIME:"; TIMER - start! ;"seconds"
' n = nn:l = 22:nn$ = "Error Occurrences =":gosub ShowNum
return

'THIS IS 1.46 TIMES THE SPEED OF THE EXP() FUNCTION AND
'DISAGREES WITH EXP() ONLY IN BITS 0 & 1 (IE BY AT MOST
'3 IN 32767)

Sigmoid:
  if x < 0 then xx = -x else xx = x
  xxx! = xx
  i = INT(xxx! / 32)
  dx = xx - i * 32
  y = SigTab(i)
  y = y + ((dx * (SigTab(i + 1) - y)) / 32)
  if x < 0 then y = -y
' w! = x
' v! = RR! / (1 + exp(-w! * k / R!)) - R!
' yy = v!
' n = abs(yy-y)
' locate 17,13 : print n
' if n > 0 then nn = nn + 1
return

ShowNum:
  n$ = str$(n) : n$ = space$(6 - len(n$)) + n$
  locate l,5: print nn$ + n$
return
```

This function exercises the look-up table function for 4 different values of k. It can also compare the execution times of the exp and the look-up method and measure the maximum error of the look-up method.

Load sigmoid.bas into a QuickBASIC environment and compile and execute it. This will exercise the QuickBASIC version of the Sigmoid function and display the graphical results. You can alter the code in order to experiment with the program.

Always 'comment-out' the display statements in the Sigmoid: subroutine when doing time trials as they are themselves time-consuming. Note that the speed advantage in BASIC between the exp function and the look-up table is nowhere near as much as it is in 'C'.

The 'C' Version

QuickBASIC is a very quick means of getting a prototype like this working and visible. However you cannot use BASIC to optimise efficiency and speed since it does not provide the almost direct control of the CPU that 'C' does. An exerciser for the 'C' version of Sigmoid() was therefore written and exercised within the Microsoft Programmer's Workbench development environment using Microsoft C 6.00 as follows.

```
#include <sys\types.h>
#include <sys\timeb.h>
#include <stdio.h>
#include <graph.h>
#include <math.h>
long PlotErr[9];              //error-count accumulators
```

```
//THE FOLLOWING TWO FUNCTIONS ARE TO BE EXERCISED

short   SigTab[1025];
#define R  32767
#define RR 65556               //65534 + 22

void SigGen(void) {
  int i;
  for(i = 0; i < 1024; i++)
  SigTab[i] = (double)(
    RR / (1 + exp( -((double)(((long)(i)) << 8)) / R)) - R
  );
  SigTab[1024] = R;
}

int Sigmoid(int x) {
  int s, y, j;
  if((s = x) < 0) x = -x;
  y = *(SigTab + (j = x >> 5));
  y += ((*(SigTab + j + 1) - y) * (x & 0x1F)) >> 5;
  if(s < 0) y = -y;
  return(y);
}

// BELOW IS THE EXERCISER FOR THE TWO FUNCTIONS ABOVE

#define XYscale 8    //right shift scaling factor
#define Yorg 174     //origin displacement from top left
#define Xorg 171     //origin displacement from top left

void ShowScale(short x1, short y1, short x2, short y2) {
  _moveto((x1 >> XYscale) + Xorg, (y1 >> XYscale) + Yorg);
  _lineto((x2 >> XYscale) + Xorg, (y2 >> XYscale) + Yorg);
}

double GetTime() {
  struct timeb TimeData;
  ftime(&TimeData);
  return((double)(TimeData.millitm) / 1000 + TimeData.time);
}

main() {
  int x, y, X, Y;
  double LookTime, ExpTime;
  _setvideomode(_VRES16COLOR);
  _settextposition(1, 1);
  printf("SIGMOID FUNCTION FOR USE IN NEURAL NETWORKS");
  _settextposition( 3,19); printf("+32767");
  _settextposition(20,19); printf("-32767");
  _settextposition(11, 3); printf("-32767");
  _settextposition(12,35); printf("+32767");
  _setcolor(7);
  ShowScale(0, -32767, 0, +32767);  //Draw the x and y axes
  ShowScale(-32767, 0, +32767, 0);  //in dull white
  _settextposition( 4,46);
  printf("The SIGMOID formula is:");
  _settextposition( 6,46);
  printf("f(x) = (2R+d)/(1+exp(-kx)) - R");
```

```
  _settextposition( 8,46);
  printf("Where R = 32767 and");
  _settextposition( 9,46);
  printf("-32767 <= x <= +32767");
  _settextposition(11,46);
  printf("`k' determines how step-like the");
  _settextposition(12,46);
  printf("function becomes. In this test");
  _settextposition(13,46);
  printf("k = 8. `k' can conveniently be a");
  _settextposition(14,46);
  printf("multiple of 2: 2,4,8,16,32,64...");
  _settextposition(16,46);
  printf("`d' is a small adjusting factor");
  _settextposition(17,46);
  printf("to make f(x) hit 32767 at exactly");
  _settextposition(18,46);
  printf("the point at which x hits 32767.");
  SigGen();     //generate the look-up table
  _setcolor(9); //Draw look-up trace in bright blue
  x = -32767;   //Draw the sigmoid curve
Label1:
  y = Sigmoid(x);
  _setpixel(Xorg + (x >> XYscale), Yorg - (y >> XYscale));
  if (x < 32767) { x++; goto Label1; }
  _settextposition(24,1);
  printf("CHECKING ACCURACY OF LOOKUP & INTERPOLATION...");
  _settextposition(6,8); printf("Error");
  _setcolor(10);  //Draw formula trace in bright green
  x = -32767; Y = 0;
Label2:
  if(x < 0) X = -x; else X = x;
  y = (double)(
    RR / (1 + exp(-((double)(((long)(X)) << 3)) / R)) - R
  );
  if (x < 0) y = -y;
  _setpixel(Xorg + (x >> XYscale), Yorg - (y >> XYscale));
  y -= Sigmoid(x);
  (PlotErr[y+2])++;  //Increment appropriate error count
  if(y > Y) Y = y;
  _settextposition(6,14);
  printf("%2d", y); //display current error amount
  if (x < 32767) { x++; goto Label2; }
  _settextposition(7,  8);
  printf("MaxErr%2d",Y);
  _settextposition(20,43);
  printf("RESULTS:");
  _settextposition(20,53);
  printf("Error = +2:  %6ld",PlotErr[4]);
  _settextposition(21,53);
  printf("Error = +1:  %6ld",PlotErr[3]);
  _settextposition(22,53);
  printf("Error =  0:  %6ld",PlotErr[2]);
  _settextposition(23,53);
  printf("Error = -1:  %6ld",PlotErr[1]);
  _settextposition(24,53);
  printf("Error = -2:  %6ld",PlotErr[0]);
  _settextposition(24,1);
  printf("CHECKING SPEED ADVANTAGE...                  ");
```

```
  x = -32767;
  ExpTime = GetTime();              //Start the timer
Label3:
  if(x < 0) X = -x; else X = x;
  y = (double)(
    RR / (1 + exp(-((double)(((long)(X)) << 3)) / R)) - R
  );
  if(x < 0) y = -y;
  if (x < 32767) { x++; goto Label3; }
  ExpTime = GetTime() - ExpTime;    //Stop the timer
  x = -32767;
  LookTime = GetTime();             //Start the timer
Label4:
  y = Sigmoid(x);
  if(x < 32767) { x++; goto Label4; }
  LookTime = GetTime() - LookTime;  //Stop the timer
  _settextposition(26,53);
  printf("Exp(x) Time:  %4.2f", ExpTime);
  _settextposition(27,53);
  printf("Lookup Time:  %4.2f", LookTime);
  _settextposition(28,53);
  printf("Speed Ratio:  %4.2f", ExpTime/LookTime);
  _settextposition(24,1);
  printf("FINISHED. HIT `RETURN' TO EXIT.\07");
  _settextposition(26,1); getchar();
  _setvideomode(_DEFAULTMODE);
}
```

This source code is contained in the file sigmoid.c and its executable version is in sigmoid.exe. Run sigmoid.exe to exercise the 'C' version of Sigmoid(), or load sigmoid.c into the Microsoft Programmer's Workbench then MAKE and RUN the program. You can modify the code yourself and experiment with it. I later upgraded to Linux where I use the gc compiler.

SigGen( )'s Coding

```
void SigGen(void) {
  #define R  32767     //maximum extent of both x and f(x)
  #define RR 65556     //65534 + 22
  int i;
  for(i = 0; i < 1024; i++)
    SigTab[i] = (double)(
      RR / (1 + exp( -((double)(((long)(i)) << 8)) / R)) - R
    );
  SigTab[1024] = R;
}
```

This code is tuned for maximum performance for k = 8. Notice particularly that RR is a little bit more than twice R. In fact it is 22 more than 2 * R. The sigmoid function computed from the exponential formula only reaches its maximum value of 32767 when x reached infinity. To avoid a progressive loss of range across the multi-layer network, x is therefore accentuated slightly so that f(x) hits 32767 precisely when x does.

The value of RR will need to be re-adjusted if you change the value of k. You will have to use something like the following to display the values of x and f(x) so that you can adjust the final few values of f(x) to intercept the maximum value of 32767 smoothly:

```
main() {
  #define R  32767   //maximum extent of both x and f(x)
  #define RR 65556   //65534 + 22
  #define K      8   //set the value of K you want here
  int x;
  for(x = 0; x < 32766; x++) {
  y = (double)(
    RR / (1 + exp( -(((double)(x)) * K) / R)) - R
  );
  printf("x = %5d     f(x) = %5d\n", x, y);
  }
}
```

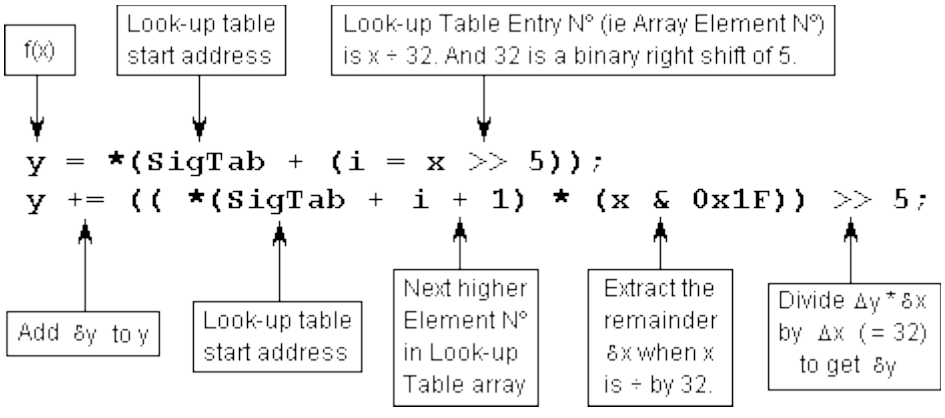
See source file siggen.c and executable siggen.exe.

Sigmoid( )'s Coding

```
int Sigmoid(int x) {
  int s, y, j;
  if ((s = x) < 0) x = -x;   //reverse sign of x if negative
  y = *(SigTab + (j = x >> 5));
  y += ((*(SigTab + j + 1) - y) * (x & 0x1F)) >> 5;
  if(s < 0) y = -y; //reverse sign of f(x) if x was negative
  return(y);
}
```

Because of its time-critical position at the very centre of the innermost loop of the Multi-layer Perceptron neural network function MLP(), great effort has been made to make Sigmoid() as fast as possible.

Thus multiplications and divisions have been replaced wherever possible by binary shifts, and remainder extraction has been replaced by logical masking. This has been implemented in Sigmoid() as shown below:



Because the above two statements will work only with positive numbers, negative values are reversed before and after the two above statements. Testing for less than zero was found to be the quickest and safest way to reverse the signs of negative values since the 80x86 CPU represents negative values as the complement of the equivalent positive values:

```
  +32767     7FFF
  +    1     0001
  -    1     FFFF
  -32767     8001
```

This does not make it easy to change the sign by bitwise logic. However, if you are using a CPU with 'sign-mantissa' arithmetic, you could improve the speed of Sigmoid() even further by coding it as follows:

```
int Sigmoid(int s) {
  int x, y, j;
  s = x & 0x7FFF;             //reverse sign of x if negative
  y = *(SigTab + (j = x >> 5));
  return(
    (s & 0x8000) |
    (y += ((*(SigTab + j + 1) - y) * (x & 0x1F)) >> 5)
  );
}
```

This won't work on machines that use complement arithmetic and masks must be compatible with the machine's word-length. A 16-bit word length is assumed above. The function becomes quite machine-dependent, but the extra speed could be worth it if you do a lot of network training runs.

Test Results

The following table shows the results produced by Sigmoid() while running in the above exerciser. It shows how many of the 65335 values of f(x) returned by Sigmoid() were in error by +2, +1, 0, −1, −2 compared with the corresponding value returned by the exp(x) formula:

	Number of Entries in the Table					
	128	256	512	1024	2048	4096
+2	4088	771	17	0	0	0
+1	16005	19241	15162	13087	10904	08639
Errors 0	17117	25511	35177	39361	43727	48257
-1	16005	19241	15162	13087	10904	08639
-2	4088	771	17	0	0	0
Max Error	±4	±2	±2	±1	±1	±1
Δx	256	128	64	32	16	8
Δy	1024	512	256	128	64	32
Δy * Δx	262144	65536	16368	4096	1024	256
Precision	32-bit	32-bit	16-bit	16-bit	16-bit	16-bit

As can be seen from the lower part of the above table, a long casting of Dy had to be used for tables with 256 entries or less otherwise the product Dy * Dx would overflow the 16-bit short register. For a 256-entry table, the 4th line of Sigmoid() therefore had to become:

```
y += ((long)((*(SigTab + i + 1) - y)) * (x & 0x7F)) >> 7;
```

This however resulted in a 20% degradation in speed as shown by the following time trial results:

	(long)(Δy)	(short)(Δy)
Using the exp(x) formula	351.80 s	351.80 s
Using the Lookup Table	001.70 s	001.37 s
Speed Ratio	206.94	256.79

Because it helped speed up the BASIC version, an attempt to preserve as much speed as possible was made by switching precision based on the value of dx as follows:

```
short Sigmoid(short s) {
  int x, y, i, dx, Dy;   //dx represents dx, Dy represents Dy
  if((x = s) < 0) x = -x;
  y = *(SigTab + (i = x >> 7));
  if((dx = x & 0x7F) > 0) {       //if we need to interpolate
    Dy = *(SigTab + i + 1) - y;   //diff between two y entries
    if (Dy < 259)
      y += (Dy * dx) >> 7;            //use 16-bit precision
    else
      y += ((long)(Dy)) * dx) >> 7;   //use 32-bit precision
  }
  if (s < 0) y = -y;
  return(y);
}
```

This however had the opposite effect due to the time taken up by the extra **if** statements.

The optimum size of look-up table was therefore 1024 giving a firm value of f(x) for increments in x of 32 with a maximum error of ±1. Nothing was gained by using larger tables since the error in the least significant bit would be present anyway from binary rounding errors on the shifts and multiplies. Besides this is only an error of 30 parts per million which is better than the inputs originating from most input sources. Smaller tables started to allow the linear interpolation errors to creep in rather rapidly.

Other Transfer Functions

The sigmoid is the ideal non-linear transfer function for the multi-layer perceptron. However, there are other kinds of neural network topologies in which different non-linear transfer functions work better. For instance, in the Radial Basis Network topology, the Gaussian (or bell curve) function is used. This function, together with a test exerciser are available in gauss.c (executable in gauss.exe) and gauss_test.c (executable in gauss_test.exe).
